Universal scaling in coupled maps

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Using a renormalization method, we study critical scaling behaviors of all period p-tuplings ($p=2,3,4,\ldots$) in two symmetrically coupled one-dimensional (1D) maps near the symmetry line. We find three (five) kinds of fixed points of a renormalization operator for the case of even (odd) p. The relevant "coupling eigenvalues" associated with coupling perturbations vary depending on the kinds of fixed point, while the relevant eigenvalue associated with scaling of the nonlinearity parameter of the uncoupled 1D maps is a common one to all the fixed points. With an example, we also confirm the renormalization results.

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Critical behaviors of period p-tuplings ($p = 2,3,4,\ldots$) were much studied in one-dimensional (1D) unimodal maps [1–8]. It was found that the asymptotic scaling behaviors of the period p-tupling sequences characterized by the orbital and parameter scaling factors, α and δ, vary depending on p. Recently the critical behavior of period doublings ($p = 2$) was also studied [9–11] in coupled 1D maps, which are used as models of coupled nonlinear oscillators such as Josephson-junction arrays, chemically reacting cells, and so on [12]. New scaling behaviors associated with coupling perturbations were found in such 1D coupled maps.

In this Brief Report we are interested in the critical behaviors of all the other higher period p-tuplings ($p = 3,4,\ldots$) in two-coupled 1D maps. We first investigate the dependence of the scaling behaviors on p using a renormalization method. It is found that in the case of even (odd) p there exist three (five) kinds of fixed points of a renormalization operator. All the fixed points have a common relevant eigenvalue δ associated with the critical scaling of the nonlinearity parameter of the uncoupled 1D maps. However, the relevant "coupling eigenvalues" (CE's) associated with coupling perturbations vary depending on the kind of fixed points. As an example, we study the scaling behavior of period triplings ($p = 3$) in dissipatively coupled 1D maps, and confirm the renormalization results. An extended version of this work including a detailed account of the renormalization results, the numerical results for the linear-coupling case, an extension to many coupled maps, and so on will be given elsewhere [13].

We consider a map T consisting of two symmetrically coupled 1D maps,

$$T: \begin{cases} x_{t+1} = F(x_t, y_t) = f(x_t) + g(x_t, y_t), \\ y_{t+1} = F(y_t, x_t) = f(y_t) + g(y_t, x_t), \end{cases} \tag{1}$$

where $f(x)$ is a 1D unimodal map with a quadratic maximum, and $g(x,y)$ is a coupling function obeying the condition $g(x,x) = 0$ for any x. This two-coupled map is invariant under the exchange of coordinates such that $x \leftrightarrow y$. The set of all points which are invariant under the exchange of coordinates forms a symmetry line $y = x$. An orbit is called an "in-phase" orbit if it lies on the symmetry line, i.e., it satisfies $x_t = y_t$ for all t. Otherwise it is called an "out-of-phase" orbit. Here we study only the in-phase orbits.

Stability of an in-phase orbit with period q is determined from the Jacobian matrix J of T^q, which is the q product of the Jacobian matrix DT of T along the orbit:

$$J = \prod_{t=1}^{q} \begin{pmatrix} f'(x_t) - G(x_t) & G(x_t) \\ G(x_t) & f'(x_t) - G(x_t) \end{pmatrix}, \tag{2}$$

where the prime denotes a derivative, and $G(x) = \partial g(x,y)/\partial y \mid_{y=x}$; hereafter, $G(x)$ will be referred to as the "reduced coupling function" of $g(x,y)$. The eigenvalues of J, called the stability multipliers of the orbit, are

$$\lambda_1 = \prod_{t=1}^{q} f'(x_t), \quad \lambda_2 = \prod_{t=1}^{q} [f'(x_t) - 2G(x_t)]. \tag{3}$$

Note that the first stability multiplier λ_1 is just that of the uncoupled 1D map and the coupling affects only the second stability multiplier λ_2, which may be called the "coupling stability multiplier." The in-phase orbit is stable only when the moduli of both multipliers are less than or equal to unity, i.e., $-1 \leq \lambda_i \leq 1$ for $i = 1,2$.

We now consider the period p-tupling ($p = 2,3,4,\ldots$) renormalization transformation $\mathcal{N}$ for a coupled map T, which is composed of the p-times iterating ($T^{(p)}$) and rescaling (B) operators:

$$\mathcal{N}(T) \equiv BT^{(p)}B^{-1}. \tag{4}$$

Here the rescaling operator B is

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$$B = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad (5)$$

because we consider only in-phase orbits.

Applying the renormalization operator $\mathcal{N}$ to the coupled map (1) n times, we obtain the n-times renormalized map T_n of the form

$$T_n : \begin{cases} x_{t+1} = F_n(x_t, y_t) = f_n(x_t) + g_n(x_t, y_t), \\ y_{t+1} = F_n(y_t, x_t) = f_n(y_t) + g_n(y_t, x_t). \end{cases} \quad (6)$$

Here f_n and g_n are the uncoupled and coupling parts of the n-times renormalized function F_n, respectively. They satisfy the following recurrence equations:

$$f_{n+1}(x) = \alpha f_n^{(p)}\left(\frac{x}{\alpha}\right), \quad (7)$$

$$g_{n+1}(x,y) = \alpha F_n^{(p)}\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) - \alpha f_n^{(p)}\left(\frac{x}{\alpha}\right), \quad (8)$$

where $f_n^{(p)} = f_n(f_n^{(p-1)}(x))$ and $F_n^{(p)}(x,y) = F_n(F_n^{(p-1)}(x,y), F_n^{(p-1)}(y,x))$. [Note that the right-hand side of Eq. (8) consists of f_n and g_n because $F_n(x,y) = f_n(x) + g_n(x,y)$.] Hence the recurrence relations (7) and (8) define a renormalization operator $\mathcal{R}$ of transforming a pair of functions (f, g), $(f_{n+1}, g_{n+1}) = \mathcal{R}(f_n, g_n)$.

A map T_c with the nonlinearity and coupling parameters set to their critical values is called a critical map:

$$T_c : \begin{cases} x_{t+1} = F_c(x_t, y_t) = f_c(x_t) + g_c(x_t, y_t), \\ y_{t+1} = F_c(y_t, x_t) = f_c(y_t) + g_c(y_t, x_t). \end{cases} \quad (9)$$

A critical map is attracted to a fixed map T^* under iterations of the renormalization transformation $\mathcal{N}$:

$$T^* : \begin{cases} x_{t+1} = F^*(x_t, y_t) = f^*(x_t) + g^*(x_t, y_t), \\ y_{t+1} = F^*(y_t, x_t) = f^*(y_t) + g^*(y_t, x_t). \end{cases} \quad (10)$$

Here (f^*, g^*) is a fixed point of the renormalization operator $\mathcal{R}$, i.e., $(f^*, g^*) = \mathcal{R}(f^*, g^*)$. Note that $f^*(x)$ is just the fixed function in the 1D map case, which varies depending on p [4,5,7,8]. Only the equation for the coupling fixed function $g^*(x,y)$ is therefore left to be solved. One trivial solution is $g^*(x,y) = 0$. In this zero-coupling case, the fixed map (10) consists of two uncoupled 1D fixed maps, which is associated with the critical behavior at the zero-coupling critical point.

However, it is not easy to directly find coupling fixed functions other than the zero-coupling fixed function $g^*(x,y) = 0$. We therefore introduce a tractable recurrence equation for a reduced coupling function $G(x) = \partial g(x,y)/\partial y|_{y=x}$. Differentiating the recurrence equation (8) for $g(x,y)$ with respect to y and setting $y = x$, we obtain a recurrence equation for $G(x)$:

$$G_{n+1}(x) = F_{n,2}^{(p)}\left(\frac{x}{\alpha}\right) \quad (11)$$

$$= F_{n,2}^{(p-1)}\left(\frac{x}{\alpha}\right)\left[f_n'\left(f_n^{(p-1)}\left(\frac{x}{\alpha}\right)\right) - 2G_n\left(f_n^{(p-1)}\left(\frac{x}{\alpha}\right)\right)\right] + f_n^{(p-1)\prime}\left(\frac{x}{\alpha}\right) G_n\left(f_n^{(p-1)}\left(\frac{x}{\alpha}\right)\right), \quad (12)$$

where the subscript 2 of F_n denotes a partial derivative with respect to the second argument. Then Eqs. (7) and (12) define a "reduced renormalization operator" $\tilde{\mathcal{R}}$ of transforming a pair of functions (f, G), $(f_{n+1}, G_{n+1}) = \tilde{\mathcal{R}}(f_n, G_n)$. We look for fixed points (f^*, G^*) of $\tilde{\mathcal{R}}$, which satisfy $(f^*, G^*) = \tilde{\mathcal{R}}(f^*, G^*)$. Here f^* is just the 1D fixed function and G^* is the reduced coupling fixed function of g^*, i.e., $G^*(x) = \partial g^*(x,y)/\partial y\,|_{y=x}$. We find three (five) solutions for G^* in the case of even (odd) p:

$$G^*(x) = 0, \quad (13a)$$

$$G^*(x) = \frac{1}{2} f^{*\prime}(x), \quad (13b)$$

$$G^*(x) = \frac{1}{2}[f^{*\prime}(x) - 1], \quad (13c)$$

$$G^*(x) = \frac{1}{2}[f^{*\prime}(x) + 1], \quad (13d)$$

$$G^*(x) = f^{*\prime}(x), \quad (13e)$$

where the solutions (13a)–(13c) exist for any p, but the solutions (13d) and (13e) exist only for odd p. The reduced fixed points (f^*, G^*) with solutions (13b)–(13e) are associated with critical behaviors at critical points other than the zero-coupling critical point. The first three solutions (13a)–(13c) have been already found for the lowest even case with $p = 2$ [11], and similarly one can show that all five solutions (13a)–(13e) exist for the lowest odd case with $p = 3$. Then, by induction, one can obtain the solutions (13) for all the other cases of higher period p-tuplings ($p = 4, 5, \ldots$) [13].

Consider an infinitesimal perturbation (h, Φ) to a fixed point (f^*, G^*) of the reduced renormalization operator $\tilde{\mathcal{R}}$. Linearizing $\tilde{\mathcal{R}}$ at the fixed point, we obtain the recurrence equation for the evolution of (h, Φ), $(h_{n+1}, \Phi_{n+1}) = \tilde{\mathcal{L}}(h_n, \Phi_n)$. A pair of perturbations of (h^*, Φ^*) is called an eigenperturbation with eigenvalue ν if $\tilde{\mathcal{L}}(h^*, \Phi^*) = \nu(h^*, \Phi^*)$. All fixed points (f^*, G^*) have a common relevant eigenvalue δ (i.e., the relevant eigenvalue for the cases of the uncoupled 1D maps) associated with the critical scaling of the nonlinearity parameter of the uncoupled 1D maps. However, the coupling eigenperturbations $(0, \Phi^*)$ with relevant CE's ν_c depend on the kinds of fixed points. The relevant CE's are associated with the critical scaling of the coupling parameter. Like the case of obtaining the reduced coupling fixed functions G^*'s, by

TABLE I. Reduced coupling fixed functions $G^*(x)$, relevant CE's ν_c, and the critical coupling stability multipliers λ_2^* in all the period p-tupling ($p = 2, 3, 4, \ldots$) cases are shown. Note that the case $G^*(x) = \frac{1}{2} f^{*\prime}(x)$ has no relevant CE's, and α and λ^* are the orbital scaling factor and the critical stability multiplier for the 1D case, respectively.

$G^*(x)$	ν_c	λ_2^*
0	α, p	λ^*
$\frac{1}{2} f^{*\prime}$	nonexistent	0
$\frac{1}{2}[f^{*\prime}(x) - 1]$	p	1
$\frac{1}{2}[f^{*\prime}(x) + 1]$	p	-1
$f^{*\prime}(x)$	α, p	$-\lambda^*$

induction, we also obtain the relevant CE's for each fixed point (f^*, G^*) in all the period p-tupling cases [13], which are listed in Table I.

In the case of a critical map (9), the stability multipliers $\lambda_{1,n}$ and $\lambda_{2,n}$ of the in-phase orbits with period $q = p^n$ $(n = 1, 2, \ldots)$ converge to the critical stability multipliers λ_1^* and λ_2^*, respectively, as the level n increases to the infinity. Since λ_1 depends only on the nonlinearity parameter, λ_1^* is always the same as the critical stability multiplier λ^* for the 1D case. However, the critical coupling stability multiplier λ_2^* depends on the reduced coupling function $G^*(x)$ as follows [13]:

$$\lambda_2^* = \lambda^* - 2G^*(\hat{x}), \tag{14}$$

where the 1D critical stability multiplier λ^* is given by $\lambda^* = f'^*(\hat{x})$, and $\hat{x}$ is the fixed point of the uncoupled 1D fixed map, i.e., $\hat{x} = f^*(\hat{x})$. Substituting G^*'s of Eq. (13) into Eq. (14), one can obtain λ_2^*'s, which are also listed in Table I.

The structure of the critical set (set of critical points) and the critical behaviors vary depending on whether the coupling function $g(x, y)$ has a leading linear term or not; a coupling is called linear or nonlinear according to its leading term. As an example, we study the period-tripling case in two dissipatively coupled 1D maps (1) with $f(x) = 1 - Ax^2$ and $g(x, y) = \frac{c}{2}[f(y) - f(x)]$, and confirm the renormalization results. Here A is the nonlinearity parameter of the uncoupled 1D map and c is the coupling parameter. Note that the dissipative coupling is a kind of nonlinear coupling.

It follows from (3) that the stability multipliers of in-phase orbits with period $q = 3^n$ $(n = 1, 2, \ldots)$ become

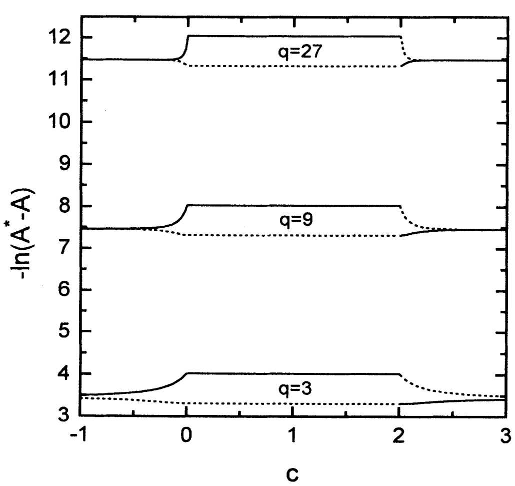

FIG. 1. Stability diagram of the in-phase orbits with period $q = 3^n$ $(n = 1, 2, 3)$ in two dissipatively coupled 1D maps. The stable region of each periodic orbit is bounded by four bifurcation curves associated with tangent and period-doubling bifurcations. Here the horizontal (nonhorizontal) solid lines denote the period-doubling bifurcation curves $\lambda_{1,n}$ $(\lambda_{2,n}) = -1$, whereas the horizontal (nonhorizontal) dashed lines denote the tangent bifurcation curves $\lambda_{1,n}$ $(\lambda_{2,n}) = 1$.

$$\lambda_{1,n} = \prod_{t=1}^{q} f'(x_t), \quad \lambda_{2,n} = (1-c)^q \lambda_{1,n}, \tag{15}$$

because the reduced coupling function for the dissipative-coupling case becomes $G(x) = \frac{c}{2} f'(x)$. The stable region of each periodic orbit of level n (period 3^n) in the space of the nonlinearity and coupling parameters is bounded by four bifurcation curves associated with tangent and period-doubling bifurcations (i.e., those curves determined by the equations, $\lambda_{i,n} = \pm 1$ for $i = 1, 2$). The stability regions for the cases of the first three levels $(n = 1, 2, 3)$ are shown in Fig. 1. Since the "height" h_n of the stability region of level n, which is defined as the length of the $c = 1$ line segment inside the stable region of level n, geometrically contracts in the limit of large n,

$$h_n \sim \delta^{-n} \quad \text{for large } n, \tag{16}$$

we choose as a vertical coordinate the quantity $-\ln(A^* - A)$ instead of A. Here δ $(= 55.247\,026\cdots)$ and A^* $(= 1.786\,440\,255\,563\,639\,354\,534\,447\cdots)$ are the relevant eigenvalue associated with the critical scaling of the nonlinearity parameter A and the accumulation point of the period-tripling sequence for the 1D case, respectively. An infinite sequence of such stability regions accumulates to a critical line connecting two ends (A^*, c_l^*) and (A^*, c_r^*), where $c_l^* = 0$ and $c_r^* = 2$, as will be seen below.

Consider two dissipatively coupled 1D maps (1) on the line $A = A^*$, in which case the reduced coupling function is given by $G(x) = \frac{c}{2} f'(x)$. By successive actions of the reduced renormalization operator $\tilde{\mathcal{R}}$ on (f, G), we obtain

$$f_n(x) = \alpha f_{n-1}^{(3)}\left(\frac{x}{\alpha}\right), \quad G_n(x) = \frac{c_n}{2} f_n'(x), \tag{17}$$

$$c_n = c_{n-1}^3 - 3c_{n-1}^2 + 3c_{n-1}, \tag{18}$$

where $f_0(x) = f(x)$, $G_0(x) = G(x)$, and $c_0 = c$. Here f_n converges to the 1D fixed function $f^*(x)$ because the nonlinearity parameter A is set to its critical value A^*.

The recurrence equation (18) for c has three fixed points c^*:

$$c^* = 0,\ 1,\ 2. \tag{19}$$

Stability of a fixed point c^* is determined by its stability multiplier μ given by $\mu = dc_n/dc_{n-1}|_{c^*}$. The fixed point at $c^* = 1$ is superstable $(\mu = 0)$, whereas the other ones at $c^* = 0, 2$ are unstable $(\mu = 3)$. The basin of attraction to the superstable fixed point becomes the open interval $(0, 2)$ because any initial c inside the interval $0 < c < 2$ converges to $c^* = 1$; all points outside the interval diverge to the plus or minus infinity. Consequently the critical set for this dissipative-coupling case is the critical line segment joining two ends $c_l^* = 0$ and $c_r^* = 2$ on the $A = A^*$ line. Inside the critical line segment all critical maps are attracted to the fixed maps with the same reduced coupling function $G^*(x) = \frac{1}{2} f^{*\prime}(x)$. These fixed maps have no relevant CE's, because the fixed point $c^* = 1$ is superstable. On the other hand, the critical map at the left end (i.e., the zero-coupling critical point) is attracted to the zero-coupling fixed map with $G^*(x) = 0$, and the critical map at the right end $(A^*, 2)$ to another fixed map with $G^*(x) = f^{*\prime}(x)$. These two fixed maps have one relevant CE $\nu_c = 3$, because the fixed points $c^* = 0, 2$ are unstable ones with stability multiplier $\mu = 3$. Substituting $c^* = 1,\ 0,\ 2$ into Eq. (15), we also obtain the critical coupling stability multipliers $\lambda_2^* = 0,\ \lambda^*,\ -\lambda^*$, respectively, where λ^* ($= -1.872\,705\cdots$) is the critical stability multiplier for the 1D case.

However, the structure of the critical set for a linear-coupling case with a leading linear term is different from that for a nonlinear-coupling case [13]. For even (odd) p, there appear three (four) kinds of fixed maps in a linear-coupling case, while only two (three) kinds of fixed maps appear in a nonlinear-coupling case (see the above example of dissipative coupling). Note also that the critical behavior near the zero-coupling point is governed by all two CE's α and p for a linear-coupling case, while it is governed by only one CE p for a nonlinear-coupling case (e.g., see the above dissipative-coupling case).

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